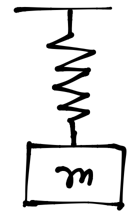
Leftovers last Friday:



Recall: A mass is attached with a spring vertically. Let u be the displacement of the mass from the equilibrium. Then the equation of motion is:

$$mu'' + \gamma u' + ku = 0$$

m — mass. γ — damping coefficient. k — spring constant.

(1) Undamped vibration: $\gamma = 0$

Natural frequency: $\omega = \sqrt{\frac{k}{m}}$.

General sol'n: $u = C_1 \cos\omega t + C_2 \sin\omega t$

Amplitude: $\sqrt{C_1^2 + C_2^2}$, Phase — angle of (C_1, C_2).

(2) Underdamped vibration: $\gamma > 0$ however not too large.

Char. eqn.: $mr^2 + \gamma r + k = 0$

Char. roots: $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

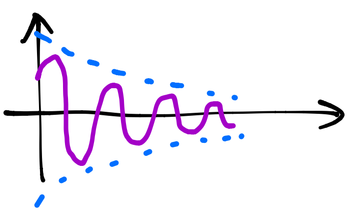
By "not too large", we mean the char. roots remain complex.

More precisely, $0 < \gamma < \sqrt{4km}$

For simplicity, write $\omega = \dfrac{\sqrt{|\gamma^2 - 4km|}}{2m}$ so the roots can be expressed as $r = -\frac{\gamma}{2m} \pm \omega i$, then the general sol'n is

$$u = C_1 e^{-\frac{\gamma}{2m}t}\cos\omega t + C_2 e^{-\frac{\gamma}{2m}t}\sin\omega t.$$

$$= \sqrt{C_1^2 + C_2^2}\, e^{-\frac{\gamma}{2m}t}\cos(\omega t - \varphi)$$



Amplitude dies out as $t \to \infty$. — Decaying Oscillation.

The resulted function is no longer periodic.

Quasi-frequency $= \omega$. Quasi-period $= \frac{2\pi}{\omega}$

Example: $mg = 4\,\text{lb}$. $mg = k \cdot 2\,\text{in} = k \cdot \frac{1}{6}\,\text{ft}$

$u(0) = 6\,\text{in} = \frac{1}{2}\,\text{ft}$. $u'(0) = 0$. $6\,\text{lb} = \gamma \cdot 3\,\text{ft/s}$.

$$\Rightarrow m = \frac{4}{32}\,\text{lb}\cdot\text{s}^2/\text{ft} = \frac{1}{8} \qquad k = \frac{4\,\text{lb}}{1/6\,\text{ft}} = 24\,\text{lb/ft}$$

$$\gamma = \frac{6\,\text{lb}}{3\,\text{ft/s}} = 2\,\text{lb}\cdot\text{s/ft}$$

Equation of motion: $\frac{1}{8}u'' + 2u' + 24u = 0$, $u(0) = \frac{1}{2}$, $u'(0) = 0$.

$$\Rightarrow u'' + 16u' + 192u = 0$$

Char. eqn. $r^2 + 16r + 192 = 0$

$r^2 + 16r + 64 - 64 + 192 = 0 \Rightarrow (r+8)^2 = -128.$

$\Rightarrow r + 8 = \pm\sqrt{-128} = \pm 8\sqrt{2}i \Rightarrow r = -8 \pm 8\sqrt{2}\,i$

General solution : $u = C_1 e^{-8t}\cos 8\sqrt{2}t + C_2 e^{-8t}\sin 8\sqrt{2}t$.

$u(0) = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2}$.

$$u'(t) = C_1\left(-8e^{-8t}\cos 8\sqrt{2}t + e^{-8t}\left(8\sqrt{2}\cdot(-1)\cdot\sin 8\sqrt{2}t\right)\right)$$
$$+ C_2\left(-8e^{-8t}\sin 8\sqrt{2}t + e^{-8t}\, 8\sqrt{2}\cos 8\sqrt{2}t\right)$$

$u'(0) = 0. \Rightarrow 0 = -8C_1 + 8\sqrt{2}\,C_2 \Rightarrow C_2 = \frac{1}{\sqrt{2}}C_1 = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

$$u(t) = \frac{1}{2}e^{-8t}\cos 8\sqrt{2}t + \frac{\sqrt{2}}{4}e^{-8t}\sin 8\sqrt{2}t$$

(3) Overdamped vibration : γ become sufficiently large.

Recall: $mu'' + \gamma u' + ku = 0$

$mr^2 + \gamma r + k = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$ distinct real roots

More precisely, $\gamma > \sqrt{4km}$.

Denote the roots by r_1, r_2, general solution.

$$u = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Note that both r_1 and r_2 are strictly negative [How?]

this means $u \to 0$ as $t \to \infty$.

Decaying exponentially.

Example: $mg = 4\text{lb}$. $mg = k\cdot 2\text{ in} = k\cdot\frac{1}{6}\text{ft}$.

$15\text{ lb} = \gamma\cdot 3\text{ft}/s$ $u(0) = 6\text{ in.} = \frac{1}{2}\text{ ft}$. $u'(0) = V$.

$\Rightarrow m = \frac{1}{8}\ \text{lb}\cdot s^2/\text{ft}$. $k = 24\ \text{lb}/\text{ft}$ $\gamma = 5\ \text{lb}\cdot s/\text{ft}$.

Eqn. of motion: $\frac{1}{8}u'' + 5u' + 24u = 0.$

$$u'' + 40u' + 192u = 0$$

Char. eqn.: $r^2 + 40r + 192 = 0$

$$r^2 + 40r + 400 - 400 + 192 = 0$$

$$(r+20)^2 = 208. \Rightarrow r + 20 = \pm\sqrt{208} = \pm\sqrt{4.4.13}$$

$$= \pm 4\sqrt{13}.$$

$$r = -20 \pm 4\sqrt{13}$$

$$u = C_1 e^{(-20+4\sqrt{13})t} + C_2 e^{(-20-4\sqrt{13})t}.$$

$u(0) = \frac{1}{2} \Rightarrow C_1 + C_2 = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2} - C_2$

$u'(0) = V \Rightarrow (-20+4\sqrt{13})C_1 + (-20-4\sqrt{13})C_2 = V.$

$$(-20+4\sqrt{13})\left(\frac{1}{2} - C_2\right) + (-20-4\sqrt{13})C_2 = V$$

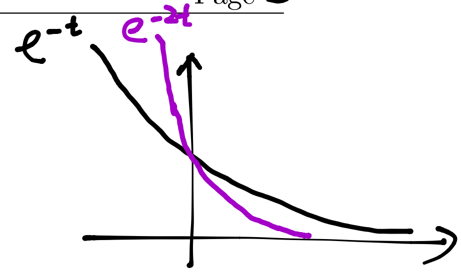
$$-10 + 2\sqrt{13} - 8\sqrt{13}\,C_2 = V \Rightarrow C_2 = \frac{V + 10 - 2\sqrt{13}}{-8\sqrt{13}} = \frac{2\sqrt{13} - 10 - V}{8\sqrt{13}}$$

$$C_1 = \frac{1}{2} - C_2 = \frac{V + 10 - 2\sqrt{13}}{8\sqrt{13}}$$

$$u = \frac{V + 10 - 2\sqrt{13}}{8\sqrt{13}} e^{(-20+\sqrt{13})t} + \frac{2\sqrt{13} - 10 - V}{8\sqrt{13}} e^{(-20-4\sqrt{13})t}$$

Mass passes through the equilibrium means $u(\tau) = 0$ for some $\tau > 0$. We know from the cond'n $u(0) > 0$, $u(\tau) = 0$ for some τ means $u(t)$ is eventually negative! [Why?]

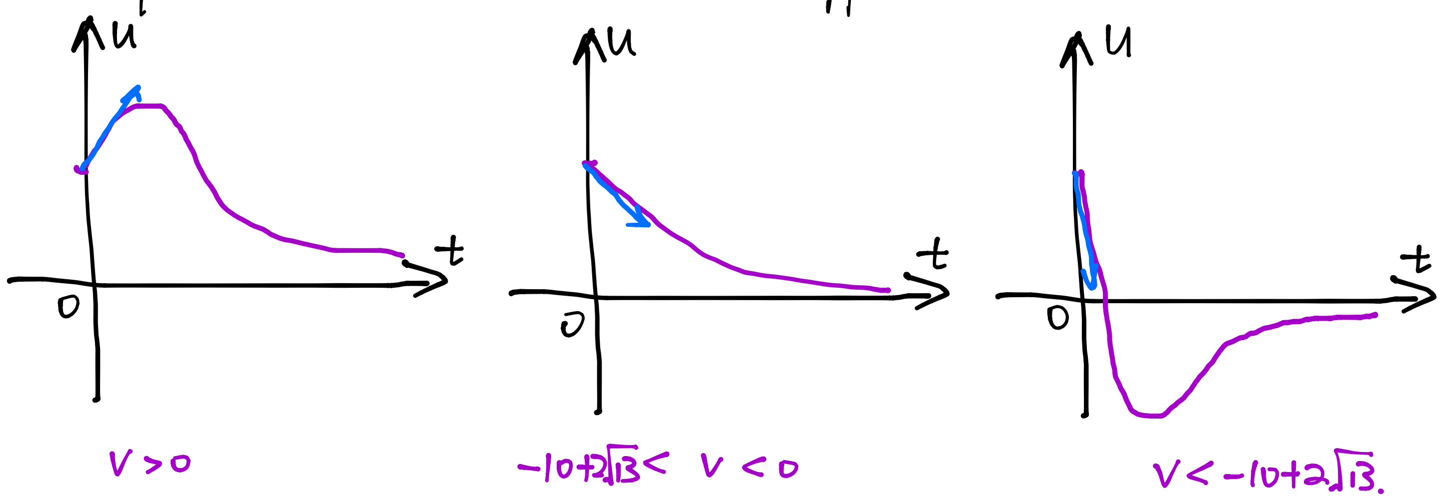
When $t \to \infty$, the solution is dominated by the first term $\frac{V+10-2\sqrt{13}}{8\sqrt{13}} e^{(-20+4\sqrt{13})t}$



If $\frac{V+10-2\sqrt{13}}{8\sqrt{13}} > 0$, then u eventually positive

$\frac{V+10-2\sqrt{13}}{8\sqrt{13}} < 0$, then u eventually negative.

In other words, if $V < -10+2\sqrt{13}$, then the mass will pass the equilibrium, i.e., overshoot happens.



5) Critically damped vibration: the moment when γ passes from under-damped to overdamped. $\gamma = \sqrt{4km}$.

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0.$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = -\frac{\gamma}{2m}, \ -\frac{\gamma}{2m}.$$

General solution: $u = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$.

$= e^{-\frac{\gamma}{2m}t}(C_1 + C_2 t)$

$t \to \infty$, $u \to 0$. i.e. mass will come back to the equilibrium.

Example: $mg = 4\text{lb}$. $k \cdot \frac{1}{8}\text{ft} = mg = 4\text{ lb}$. $u(0) = 6\text{in.} = \frac{1}{2}\text{ft}$ $u'(0)=0$.

$12 = \gamma \cdot 3\text{ft}/\text{s}$.

$\Rightarrow m = \frac{4}{32} = \frac{1}{8}$. $k = 32$, $\gamma = 4$.

$\frac{1}{8}u'' + 4u' + 32 = 0$

$u'' + 32u' + 256u = 0$.

char. eqn. $r^2 + 32r + 256 = 0 \Rightarrow (r+16)^2 = 0 \Rightarrow r = -16, -16$.

Gen. sol'n: $u = C_1 e^{-16t} + C_2 t e^{-16t}$.

$u(0) = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2}$.

$u'(0) = 0 \Rightarrow u'(t) = -16C_1 e^{-16t} + C_2(1 \cdot e^{-16t} + t \cdot (-16)e^{-16t})$

$u'(0) = -16C_1 + C_2 = 0. \Rightarrow C_2 = 16C_1 = 8$.

$u(t) = \frac{1}{2}e^{-16t} + 8te^{-16t}$ $t \to \infty$. $u(t) \to 0$.

Will u pass through the equilibrium?

$u(\tau) = 0$. $\frac{1}{2}e^{-16\tau} + 8\tau e^{-16\tau} = 0 \Rightarrow \frac{1}{2} + 8\tau = 0 \Rightarrow \tau = -\frac{1}{16} < 0$

It won't pass through the equilibrium after it's released.

In both critically damped case and overdamped case, no oscillation will be observed. The mass goes to the equilibrium real quick.

Review: Second order linear homogeneous equation

$$ay'' + by' + cy = 0$$

Char. eqn. $ar^2 + br + c = 0$ by trying $y = e^{rt}$

char. roots: r_1, r_2

I. $r_1 \neq r_2$ real. $e^{r_1 t}, e^{r_2 t}$ sol'ns. linearly independent

Gen. sol'n: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ by principle of superposition

II. $r_1 \neq r_2$ complex. $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$. $\tilde{y} = e^{(\alpha + i\beta)t}$ cplx. sol'n.

Gen. sol'n $y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$. by finding $\mathrm{Re}\,\tilde{y}$ & $\mathrm{Im}\,\tilde{y}$.

III. $r_1 = r_2 = r$ real. Gen. sol'n $y = C_1 e^{rt} + C_2 t e^{rt}$.

by variation of parameter for $\underset{u(t)}{C} e^{rt}$

Euler's equation: $ax^2y'' + bxy' + cy = 0$. $x > 0$.

Idea: Try $y = x^r$. $y' \sim x^{r-1}$. $xy' \sim x^r$

$y'' \sim x^{r-2}$ $x^2y'' \sim x^r$.

$ax^2(r(r-1)x^{r-2}) + bx(rx^{r-1}) + cx^r = ar(r-1)x^r + brx^r + cx^r = 0$

Char. eqn. $ar(r-1) + br + c = 0$

Case I: $r_1 \neq r_2$ real. x^{r_1}, x^{r_2} sol'n. $W(t^{r_1}, t^{r_2}) \neq 0$.

Gen. sol'n: $y = C_1 x^{r_1} + C_2 x^{r_2}$.

Example: $x^2y'' + 4xy' + 2y = 0$ $x > 0$

Char. eqn.: $r(r-1) + 4r + 2 = 0$.

$r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1, r_2 = -2$

Gen. sol'n: $y = C_1 x^{-1} + C_2 x^{-2}$

HW: 1c, 2a HW 11, 3, 4, 5.

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